Propagation in Twisted Square Waveguide

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Abstract—The problem of propagation of TE modes in twisted rectangular waveguides has been solved except for the case where one of the propagating modes becomes degenerate. The purpose of this paper is to show how to obtain a solution for degenerate modes in a twisted rectangular waveguide, with emphasis on the particular case of the square waveguide, for which the lowest order mode is degenerate. It is shown that the propagation constant for the twisted square waveguide can be expressed as an asymptotic series, the first term being the propagation constant for a straight square waveguide and the first order correction term being of order $1/L$ where L is the distance in which the guide makes one full rotation. The propagation constant for a nondegenerate mode in a twisted rectangular waveguide, on the other hand, can be expressed in a similar manner except that the first-order correction term is of order $1/L^2$. Some comments are offered on the nature of the transition when the propagating mode is almost degenerate.

I. INTRODUCTION

THE PROBLEM of propagation of nondegenerate TE-modes in twisted rectangular waveguides has been solved [1, pp. 96–101], and shows that for a narrow rectangular waveguide the polarization of the dominant mode twists as the guide twists. As the guide becomes more and more square an increased coupling to a cross-polarized dominant mode can be seen. In the limit as the rectangular guide becomes square, the nondegenerate solution breaks down. The square guide is degenerate in the respect that there is a cross-polarized dominant mode in addition to the normally polarized dominant mode which must be explicitly taken into consideration at the beginning of the solution by starting with two coupled modes similar to what has been done for the curved circular waveguide [1, pp. 105–111]. When the solution is found it is seen to have a quite different behavior from that of the narrow rectangular waveguide.

The square waveguide represents the case where the lowest order mode becomes degenerate. It is treated in detail here, but it will be shown that any degenerate propagating mode can be treated in the same way.

II. THE WAVE EQUATION

The twisted square guide is shown in Fig. 1. The wave equation for a twisted coordinate system X, Y, Z has been found to be [1, pp. 96–101]
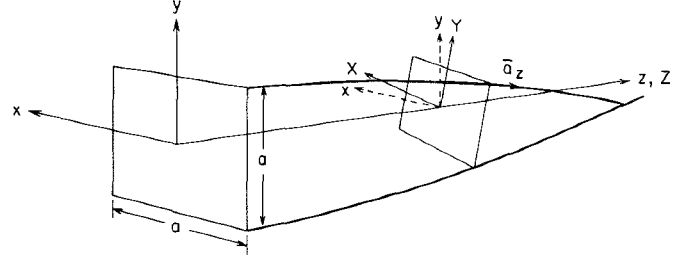
Fig. 1. Twisted square waveguide and helical coordinate system.

$$(1+p^2Y^2)\frac{\partial^2\Psi}{\partial X^2} + (1+p^2X^2)\frac{\partial^2\Psi}{\partial Y^2}$$
$$+ \frac{\partial^2\Psi}{\partial Z^2} + k^2\Psi + 2p\left(Y\frac{\partial^2\Psi}{\partial X\partial Z} - X\frac{\partial^2\Psi}{\partial Y\partial Z}\right)$$
$$- p^2\left(X\frac{\partial\Psi}{\partial X} + Y\frac{\partial\Psi}{\partial Y} + 2XY\frac{\partial^2\Psi}{\partial X\partial Y}\right) = 0 \quad (1)$$

where

$$X = x\cos(pz) + y\sin(pz)$$
$$Y = y\cos(pz) - x\sin(pz) \quad (2)$$
$$Z = z$$
$$k = 2\pi/\lambda$$
$$p = 2\pi/L \quad (3)$$

x, y, z are the coordinates in a fixed rectangular coordinate system and L is the distance in which the twisted guide makes one complete rotation. For the TE modes the electric field can be expressed as [1, pp. 96–101]

$$E_X = \frac{\partial\Psi}{\partial Y}$$
$$E_Y = -\frac{\partial\Psi}{\partial X}$$
$$E_Z = 0. \quad (4)$$

The boundary conditions for the square guide are

$$E_X = \frac{\partial\Psi}{\partial Y} = 0, \quad \text{at } Y = \pm a/2$$
$$E_Y = \frac{\partial\Psi}{\partial X} = 0, \quad \text{at } X = \pm a/2$$
$$E_Z = 0, \quad \text{on the guide surface.}$$

This last condition is automatically satisfied by (4).   (5)

III. Zeroth Order Solution

We shall seek to construct a solution as was done in the case of the rectangular guide [1, pp. 96–101] by assuming a solution of the wave equation for Ψ of the form

$$\Psi = \exp(-j\beta Z)(\Psi_0 + p\Psi_1 + p^2\Psi_2 + \cdots) \qquad (6)$$

where

$$k^2 - \beta^2 = \frac{\pi^2}{a^2}(1 + A_1 p + A_2 p^2 + \cdots). \qquad (7)$$

Substituting (6) and (7) into the wave equation (1) and examining the terms of the order p^0, the following equation is obtained:

$$\frac{\partial^2\Psi_0}{\partial X^2} + \frac{\partial^2\Psi_0}{\partial Y^2} + \frac{\pi^2}{a^2}\Psi_0 = 0. \qquad (8)$$

Solving for Ψ_0 and applying the boundary conditions of (5) it can be seen that there are two solutions, corresponding to the TE_{01} mode and the TE_{10} mode. Therefore, the following expression can be written for Ψ_0:

$$\Psi_0 = \sin\left(\frac{\pi}{a}X\right) + \alpha\sin\left(\frac{\pi}{a}Y\right) \qquad (9)$$

with α a parameter to be determined.

When we have solved for α and A_1 we find that there are two distinct modes, resembling the sum and difference of the TE_{01} and TE_{10} modes in quadrature. These two modes will propagate at different velocities and therefore, just as is found for the curved circular waveguide [1, pp. 105–111], [2, pp. 58–68], the combination of these two modes at any cross section will involve a phase change depending on the amount of twist in the guide. At certain critical points the phase change can be 90°, 180°, or 360°, causing a TE_{01} type mode to change into a TE_{10} type mode and back again as it propagates down the guide. The solution no longer propagates more or less unchanged as it does for the narrow rectangular twisted waveguide.

IV. First-Order Solution

Substituting (6) and (7) into the wave equation (1) and retaining only the terms of order p the following is obtained:

$$\frac{\partial^2\Psi_1}{\partial X^2} + \frac{\partial^2\Psi_1}{\partial Y^2} + \frac{\pi^2}{a^2}\Psi_1 + \frac{\pi^2}{a^2}A_1\left(\sin\left(\frac{\pi}{a}X\right) + \alpha\sin\left(\frac{\pi}{a}Y\right)\right)$$
$$- 2jk'\frac{\pi}{a}Y\left(\cos\frac{\pi}{a}X\right) + 2jk'\frac{\pi}{a}\alpha X\cos\left(\frac{\pi}{a}Y\right) = 0 \qquad (10)$$

with

$$k' = k\left(1 - \frac{\lambda^2}{4a^2}\right)^{1/2}.$$

The most general solution satisfying the boundary conditions is

$$\Psi_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\cos\left[\left(X - \frac{a}{2}\right)\frac{m\pi}{a}\right]\cos\left[\left(Y - \frac{a}{2}\right)\frac{n\pi}{a}\right]. \qquad (11)$$

Substituting this into (10) and applying the orthogonality properties of the cosines we get

$$(m^2 + n^2 - 1)A_{mn} = \frac{\hat{\epsilon}_m\hat{\epsilon}_n}{a^2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}$$
$$\cdot\cos\left[\left(X - \frac{a}{2}\right)\frac{m\pi}{a}\right]\cos\left[\left(Y - \frac{a}{2}\right)\frac{n\pi}{a}\right]$$
$$\cdot\left\{A_1\left(\sin\left(\frac{\pi}{a}X\right) + \alpha\sin\left(\frac{\pi}{a}Y\right)\right)\right.$$
$$\left. - 2jk'\frac{a}{\pi}Y\cos\left(\frac{\pi}{a}X\right) + 2jk'\frac{a}{\pi}\alpha X\cos\left(\frac{\pi}{a}Y\right)\right\}dX\,dY \qquad (12)$$

where

$$\hat{\epsilon}_0 = 1$$
$$\hat{\epsilon}_m = 2, \qquad m > 0.$$

It can be seen that when $m = 0$, $n = 1$ or when $m = 1$, $n = 0$, the left-hand side of (12) goes to zero. When this occurs there are only two unknowns left in (12), which can, therefore, be solved for A_1 and α, and give

$$\alpha = \pm j \qquad (13)$$
$$A_1 = -16jk'\alpha a^2/\pi^4 = \pm 16k'a^2/\pi^4. \qquad (14)$$

To assure uniqueness we require that Ψ_1 be linearly free of Ψ_0; Ψ_2 be linearly free of Ψ_1 and Ψ_0; etc., as was done with the twisted rectangular waveguide [2, pp. 26–35]. Therefore, solving (12) for A_{mn} and applying the uniqueness considerations, we find that

$$\Psi_1 = \sum_{m=0}^{\infty}{}' \sum_{n=0}^{\infty}{}' A_{mn}\cos\left[\left(X - \frac{a}{2}\right)\frac{m\pi}{a}\right]\cos\left[\left(Y - \frac{a}{2}\right)\frac{n\pi}{a}\right] \qquad (15)$$

where

$$A_{mn} = 0, \qquad m = 0, n = 0$$
$$= -8jk'\frac{a^2}{\pi^4}\frac{1 - \cos n\pi}{n^2(n^2 - 1)}, \qquad m = 0, n > 1$$
$$= \mp 8jk'\frac{a^2}{\pi^4}\frac{1 - \cos m\pi}{m^2(m^2 - 1)}, \qquad m > 1, n = 0$$
$$= 8jk'\frac{a^2}{\pi^4}\left[\frac{(1 + \cos m\pi)(1 - \cos n\pi)}{n^2(1 - m^2)}\right.$$
$$\left. \mp j\frac{(1 - \cos m\pi)(1 + \cos n\pi)}{m^2(1 - n^2)}\right]\frac{1}{m^2 + n^2 - 1}, \qquad m > 1, n > 1. \qquad (16)$$

The primes on the summations in (15) denote that the $m = 1$ and $n = 1$ terms are omitted.

V. Second-Order Solution

The second-order equation will only be solved far enough to determine the value of A_2 in (7). Substituting (6) and (7) into the wave equation (1) and retaining the terms of order p^2, the following equation is obtained:

$$\frac{\partial^2\Psi_2}{\partial X^2} + \frac{\partial^2\Psi_2}{\partial Y^2} + \frac{\pi^2}{a^2}\Psi_2 = F(X, Y) \qquad (17)$$

where

$$F(X,Y) = \frac{\pi^2}{a^2}(Y^2 - A_2)\sin\left(\frac{\pi}{a}X\right) \pm j\frac{\pi^2}{a^2}(X^2 - A_2)\sin\left(\frac{\pi}{a}Y\right)$$

$$+ \frac{\pi}{a}\left(X \mp 16j\frac{1}{\pi^2}Y\right)\cos\left(\frac{\pi}{a}X\right)$$

$$+ \frac{\pi}{a}\left(\pm jY - \frac{16}{\pi^2}X\right)\cos\left(\frac{\pi}{a}Y\right) \mp 16k'\frac{1}{\pi^2}\Psi_1 + 2jk'$$

$$\left(Y\frac{\partial \Psi_1}{\partial X} - X\frac{\partial \Psi_1}{\partial Y}\right). \quad (18)$$

Once again, the most general solution satisfying the boundary conditions is

$$\Psi_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}\cos\left[\left(X - \frac{a}{2}\right)\frac{m\pi}{a}\right]\cos\left[\left(Y - \frac{a}{2}\right)\frac{n\pi}{a}\right] \quad (19)$$

where, on applying the orthogonality conditions, it is found that

$$(m^2 + n^2 - 1)B_{mn} = \frac{\hat{\epsilon}_m\hat{\epsilon}_n}{a^2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} F(X,Y)$$

$$\cdot \cos\left[\left(X - \frac{a}{2}\right)\frac{m\pi}{a}\right]\cos\left[\left(Y - \frac{a}{2}\right)\frac{n\pi}{2}\right] dX\,dY. \quad (20)$$

When $m = 0, n = 1$ or when $m = 1, n = 0$, the left-hand side of (20) goes to zero. We can see by inspection of $F(X,Y)$ that there is only one unknown in it and therefore two equations and one unknown will result when $m = 0, n = 1$ or $m = 1, n = 0$. Fortunately, it turns out that both equations give te same result. Solving for A_2 it is found that

$$A_2 = \frac{a^2}{12} + \frac{a^2}{2\pi^2} + S \quad (21)$$

where

find that

$$\beta^2 = k'^2 - \frac{\pi^2 p}{a^2}A_1 - \frac{\pi^2 p^2}{a^2}A_2 \quad (25)$$

where A_1, A_2, and p are given in (14), (21), and (3), respectively. An alternative form of (25) is

$$\frac{1}{\lambda_T^2} = \frac{1}{\lambda_g^2} \mp \frac{1}{L}\frac{16}{\pi^2\lambda_g} - \frac{1}{L^2}\left\{\frac{1}{2} + \frac{\pi^2}{12} - \frac{128}{\pi^4}\right.$$

$$\left. + \frac{a^2}{\lambda_g^2}\left(\frac{2\pi^2}{3} + \frac{364}{3\pi^2} - \frac{1600}{\pi^4} - \frac{128}{\pi^4}g\right)\right\} \quad (26)$$

where λ_T is the wavelength in twisted square guide, the dominant wavelength in straight guide λ_g is

$$\lambda_g = \frac{\lambda}{(1 - \lambda^2/4a^2)^{1/2}}$$

and

$$g = \sum_{m=2}^{\infty} \frac{\pi(1 + \cos m\pi)(1 + 2m^2)}{2(m^2 - 1)^{9/2}}\tanh\left(\frac{\pi}{2}\sqrt{m^2 - 1}\right).$$

Numerically it is found that $g = 0.063774$, $\pi = 0.20035$. L = the distance in which the guide makes one full rotation.

A plot of $A_2\pi^2/a^2$ versus a/λ is given in Fig. 2.

VI. CHANGE IN CUTOFF FREQUENCY

We know that as the distance in which the guide makes one full rotation approaches infinity, the wavelength at the cutoff frequency will approach $2a$. Therefore, we can assume that $\Delta\lambda = (2a - \lambda_c)$, where λ_c is the wavelength at cutoff frequency in the twisted guide, will be small. An equation for the cutoff frequency can be written in terms of powers of $\Delta\lambda$, and when this has been done it can be seen that $\Delta\lambda$ is of order $1/L^2$. Therefore, retaining all

$$S = -128\frac{a^2}{\pi^6} + 64k'^2\frac{a^2}{\pi^8}\sum_{m=0}^{\infty}{}'\sum_{n=2}^{\infty}\frac{\hat{\epsilon}_m(1 + \cos m\pi)(1 - \cos n\pi)\left[m^2(1 - m^2) + n^2(1 + m^2)\right]}{n^4(m^2 - 1)^3(n^2 + m^2 - 1)}. \quad (22)$$

The doubly infinite series in (22) can be partially summed to give

$$A_2 = \frac{a^2}{\pi^2}\left\{\frac{1}{2} + \frac{\pi^2}{12} - \frac{128}{\pi^4} + \frac{a^2k'^2}{4\pi^2}\right.$$

$$\left. \cdot \left(\frac{2\pi^2}{3} + \frac{364}{3\pi^2} - \frac{1600}{\pi^4} - \frac{128}{\pi^4}g\right)\right\} \quad (23)$$

where

$$g = \sum_{m=2}^{\infty}\frac{\pi(1 + \cos m\pi)(1 + 2m^2)}{2(m^2 - 1)^{9/2}}\tanh\left(\frac{\pi}{2}\sqrt{m^2 - 1}\right). \quad (24)$$

This series is extremely rapidly convergent ($\sim 1/m^7$).

We can now express the propagation coefficient in the twisted square guide to order p^2. Solving (7) for β^2 we
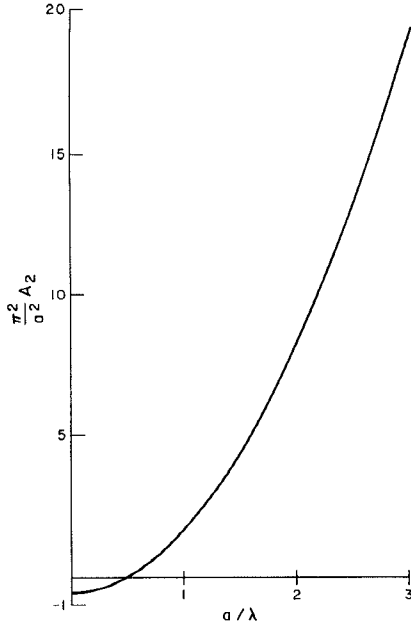
terms of order $1/L^2$ or lower we find that the cutoff frequency of the twisted square guide increases, approximately in the ratio of

$$1 + \frac{a^2}{L^2}\left[1 + \frac{\pi^2}{6} \pm \frac{16}{\pi^2}\left(2 + \frac{\pi^2}{3} - \frac{256}{\pi^4}\right)^{1/2}\right] : 1. \quad (27)$$

The plus sign in (27) gives the ratio for the mode with rotating polarization in the same direction as the guide twist ($\alpha = +j$), while the minus sign gives the ratio for the mode with the polarization rotating in the opposite direction of the twist ($\alpha = -j$).

VII. HIGHER ORDER DEGENERACIES

In Fig. 1 if we replace the waveguide dimension in the Y direction by $(2q + 1)a$, where q is an integer, we can

Fig. 2.   Graph of $A_2\pi^2/a^2$ versus a/λ.

solve for the propagation at a higher order degeneracy. The boundary conditions can now be written as

$$\frac{\partial \Psi}{\partial Y} = 0, \qquad \text{at } Y = \pm (2q+1)\frac{a}{2}$$

$$\frac{\partial \Psi}{\partial X} = 0, \qquad \text{at } X = \pm \frac{a}{2}. \qquad (28)$$

Solving for α and A_1 in the same way as was done for the square waveguide we find that, to first order in $1/L$,

$$\frac{1}{\lambda_T^2} = \frac{1}{\lambda_g^2} \pm \frac{a}{b}\frac{1}{L}\frac{16}{\pi^2 \lambda_g} \qquad (29)$$

where

$$b = (2q+1)a, \qquad q = 0,1,2,\cdots.$$

It can be seen after solving for Ψ_0 that there are two distinct modes, resembling the sum and difference of the TE_{10} and $TE_{0(2q+1)}$ modes in quadrature, where q is any nonnegative integer,

$$\Psi_0 = \sin\left(\frac{\pi}{a}X\right) \pm j \sin\frac{(2q+1)\pi Y}{b} = \sin\left(\frac{\pi}{a}X\right) \pm j \sin\left(\frac{\pi}{a}Y\right). \qquad (30)$$

When $q=0$ then $a=b$, and (29) reduces to (26) (to first order). Similarly, (30) reduces to (9).

VIII.   The Almost Degenerate Guide

The previously outlined method does not lend itself to solving the problem of the almost degenerate guide. The method for the square guide will only work exactly at a degeneracy. The solution for the rectangular guide works well when one dimension is quite a bit larger than, and not an odd integer submultiple of, the other. As the guide dimensions move closer and closer to a degeneracy the rectangular solution requires a larger and larger L for convergence and it blows up altogether when the guide

becomes degenerate. Therefore this asymptotic method yields no useful solution for a twisted guide near (but not exactly at) a degeneracy.

A similar type of difficulty can be seen in the problem of a junction between straight and curved rectangular guide [2, pp. 44–57] near the cutoff of higher order modes. It also appears that the solution of TM modes propagating in a twisted rectangular waveguide [3] similarly fails near a degeneracy, though exactly at a degeneracy the method of this paper should be applicable.

We would like to have a solution which would behave like the rectangular solution [1, pp. 96–101] when one dimension is quite a bit larger than, and not an odd integer submultiple of, the other and would behave like the leading term in the degenerate guide solution when one dimension is an odd integer multiple of the other. A possible form of solution can be constructed from the knowledge of our existing results. However, this possible solution is not unique and is shown here only to give some idea as to what form the solution might take in the case of the almost degenerate guide.

The speculated structure for the solution, which joins the rectangular and square formulas through a smooth transition is

$$\frac{1}{\lambda_T^2}$$

$$= \frac{1}{\lambda_g^2} - \frac{16}{C\pi^2}\left\{\sqrt{1 + \frac{1}{L^2}\left[\left(\frac{aC}{b\lambda_g}\right)^2 + \frac{\pi^2 C}{16}\left(1 + \frac{\pi^2 b^2}{6a^2}\right)\right]} - 1\right\} \qquad (31)$$

where

$$\lambda_g = \frac{\lambda}{\left(1 - \lambda^2/4a^2\right)^{1/2}}$$

$$C = \frac{b^2\pi^2}{8}\left[6 + \pi^2 - \frac{\pi^2 b^2}{3a^2} - \frac{2ag(\phi)}{\pi^3 b}\right]$$

$$g(\phi) = 16.902 + 128\tan\phi$$

$$-4\sum_1^\infty \frac{m^3 - (m^2 + 1/8)(m^2 - 1/4)^{1/2}\tanh\left[2\phi(m^2 - 1/2)^{1/2}\right]}{(m^2 - 1/4)^5}$$

and

$$\phi = \frac{\pi b}{2a}.$$

So long as $b \neq (2q+1)a$, so that $\phi \neq (2q+1)\pi/2$, where q is a nonnegative integer, the square root in (31) can be expanded in powers of $1/L^2$, to give the known form for λ_T. But as $b \to (2q+1)a$, $|\tan\phi| \to \infty$ and $|C|$ becomes large, so that the expansion becomes unavailable. Exactly at $b = (2q+1)a$, $|C| \to \infty$ and the square root reduces to $\pm aC/bL\lambda_g$, giving the first correction term in $1/L$ in (29). It is clear, therefore, how some such structure as (31) can bridge the two forms, but it is much more difficult to see how to develop a suitable expansion parameter therefrom. It should be stressed that no particular credence can

be claimed for the form in (31), and that the purpose of this section is to indicate the special nature of the difficulties associated with the near-degenerate cases, and why the attempted expansion runs into trouble.

X. CONCLUSIONS

With the publication of this paper (and also a forthcoming paper dealing with propagation of the TM modes [3]) we can say that the problem of propagation in a twisted rectangular guide is complete and well understood with the exception that there is still work that needs to be done to obtain a valid and useful expansion in the neighborhood of, but not exactly at, a degeneracy.

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[3] "The Propagation of TM Modes in Twisted Rectangular Waveguides," to be published.

Dispersion Relations for Comb-Type Slow-Wave Structures

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Abstract—Asymptotically accurate dispersion relations for slow eigenwaves of a dense comb placed on the wall of a parallel-plate waveguide are given in closed form. The equations can be easily resolved numerically.

An analysis of dispersion relations for combs, based on the above-mentioned equations, has advantages over commonly used methods because of the simplicity of the necessary calculations and clarity of results.

I. INTRODUCTION

DISPERSION relations for the widely used comb-type slow-wave structures usually are obtained through rather complicated computations [1], [2], [3]. In this communication asymptotically accurate dispersion relations for slow eigenwaves of a dense comb placed on the wall of a parallel-plate waveguide are given in closed form. The equations can be easily resolved numerically. If the ratio of the light velocity c to that of the slow wave v is not too small (e.g., $c/v \gg 2.5$), explicit formulas for the wavelength as a function of the phase shift can be obtained.

An analysis of dispersion relations for combs, based on the above-mentioned equations, has advantages over commonly used methods because of the simplicity of the necessary calculations and the clarity of results.

II. BASIC RELATIONS

Consider a comb placed on the wall of a parallel-plate waveguide (Fig. 1), where d is the period, h the groove depth, δ the groove width, and A the spacing between tops of the lamellas and the upper waveguide wall. We will also use the following notation: λ for the free-space wavelength, ω the circular frequency, $k = 2\pi/\lambda = \omega/c$ the wavenumber, $\kappa = d/\lambda = kd/2\pi$, $\theta = \delta/d$, $\nu = \kappa\theta \tan kh$, β the phase constant of the slow wave, $\alpha = \sqrt{\beta^2 - k^2}$ the transverse wavenumber of the slow wave, $b = \beta d/2\pi$ (where βd is the phase shift over one period), and $a = \alpha d/2\pi$.

For the case of $A = \infty$, open comb, the dispersion equation for the TM slow wave (with nonzero components E_x, E_y, and H_z) was obtained in [4, eq. (16)] with an assumption that terms of the order $\exp(-2\pi h/\delta)$ and κ^2 could be neglected (indeed, these values are generally very small in real slow-wave structures). Through some laborious and sophisticated calculations the author has succeeded in obtaining an explicit expression for the integral [4, eq. 1(b)] from (the derivation has been omitted here). With the aid of this formula and an additional assumption, i.e., $\exp(-2\pi A/d) \ll 1$ (which holds in most cases), we can now obtain a closed form of the dispersion equation for slow waves in a comb placed in a waveguide.